THE ISOPERIMETRIX IN THE DUAL BRUNN-MINKOWSKI THEORY

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ABSTRACT. We introduce the dual isoperimetrix which solves the isoperimetric problem in the dual Brunn-Minkowski theory. We then show how the dual isoperimetrix is related to the isoperimetrix from the Brunn-Minkowski theory.

1. INTRODUCTION AND STATEMENT OF MAIN RESULTS

A *definition of volume* is a way of measuring volumes on finite-dimensional normed spaces, or, more generally, on Finsler manifolds. Roughly speaking, a definition of volume μ associates to each finite-dimensional normed space $(V, \|\cdot\|)$ a norm on $\Lambda^n V$, where $n = \dim V$. We refer to [3, 28] and Section 2 for more information. The best known examples of definitions of volume are the *Busemann volume* μ^b, which equals the Hausdorff measure, the *Holmes-Thompson volume* μ^{ht}, which is related to symplectic geometry and *Gromov's mass** μ^{m*}, which thanks to its convexity properties is often used in geometric measure theory.

To each definition of volume μ may be associated a dual definition of volume μ^*. For instance, the dual of Busemann's volume is Holmes-Thompson volume and vice versa. The dual of Gromov's mass* is Gromov's mass, which however lacks good convexity properties and is used less often.

Given a definition of volume μ and a finite-dimensional normed vector space V, there is an induced $(n-1)$-density $\mu_{n-1} : \Lambda^{n-1}V \to \mathbb{R}$. Such a density may be integrated over $(n-1)$-dimensional submanifolds in V.

Given a compact convex set $K \subset V$, we let

$$A_\mu(K) := \int_{\partial K} \mu_{n-1}$$

be the $(n-1)$-dimensional surface area of K.

If the boundary ∂K is smooth, each tangent space $T_p \partial K \subset V$ carries the induced norm and ∂K is a Finsler manifold. In the general case, one may make sense of the integral by using Alexandrov's theorem [1].

The definition of volume μ is called *convex*, if for compact convex bodies $K \subset L$, we have $A_\mu(K) \le A_\mu(L)$. There are many equivalent ways of defining convexity of volume definitions, we refer to [3] for details. The above mentioned three examples are convex.

Given a convex definition of volume and an n-dimensional normed space V with unit ball B, there is a unique centrally symmetric compact convex body $\mathbb{I}_\mu B$ such that

$$A_\mu(K) = nV(K[n-1], \mathbb{I}_\mu B), \quad K \in \mathcal{K}(V).$$

Here V denotes the mixed volume and $\mathcal{K}(V)$ stands for the space of compact convex bodies in V. $\mathbb{I}_\mu B$ is called the isoperimetrix [3]. It was introduced in 1949 by Busemann [7] and has applications in crystallography [27, 29] and in geometric measure theory [5, 20, 26].

As its name indicates, the isoperimetrix is related to isoperimetric problems. More precisely, Busemann [7] showed that among all compact convex bodies of a given, fixed volume, a homothet of the isoperimetrix has minimal surface area.

The isoperimetrices of the above mentioned examples of definitions of volumes are related to important concepts from convex geometry. For Busemann's definition of volume, we have $\mathbb{I}_{\mu^b} B = \omega_{n-1}(IB)^\circ$, where $IB \subset V^*$ is the *intersection body* of B, $I^\circ B := (IB)^\circ \subset V$ its polar body, and ω_{n-1} the volume of the $(n-1)$-dimensional (Euclidean) unit ball. For the Holmes-Thompson volume, we have $\mathbb{I}_{\mu^{ht}} B = \frac{1}{\omega_{n-1}} \Pi(B^\circ)$, where Π denotes the *projection body*. It was shown recently by Ludwig [16] that the Holmes-Thompson surface area can be uniquely characterized by a valuation property. The isoperimetrix for Gromov's mass* is a dilate of the *wedge body* of B.

In the present paper, we introduce a *dual isoperimetrix* which belongs to the dual Brunn-Minkowski theory. In the dual Brunn-Minkowski theory, the natural setting is that of star bodies (i.e. compact, star shaped bodies containing the origin in their interior, with continuous radial function). The Minkowski sum of convex bodies is replaced by the radial sum and mixed volumes by dual mixed volumes. The dual Brunn-Minkowski theory was developed by Lutwak [17, 18] and plays a prominent role in modern convexity.

Let us describe our main results. The space of star bodies with smooth radial function in a finite-dimensional vector space V is denoted by $\mathcal{S}^\infty(V)$. Let $S \in \mathcal{S}^\infty(V)$ and $p \in \partial S$. We will write $\langle p \rangle := \mathbb{R} \cdot p$ for the line generated by p. The tangent space $T_p \partial S$ is naturally identified with the quotient space $V/\langle p \rangle$ and inherits the quotient norm. Therefore ∂S is in a natural way a Finsler manifold. Note that this metric is not the induced metric in general. It was (to the best of our knowledge) first studied in a recent paper by Faifman [9].

Let μ be a definition of volume, V an n-dimensional normed vector space and M a smooth $(n-1)$-dimensional submanifold such that $p \notin T_p M$ for all $p \in M$. Then $T_p M \cong V/\langle p \rangle$ can be endowed with the quotient norm and we obtain an $(n-1)$-density $\tilde{\mu}_{n-1} : \Lambda^{n-1} TM \to \mathbb{R}$. In the particular case where $M = \partial S$ for some smooth star-shaped set S, we denote by

$$\tilde{A}_\mu(S) := \int_{\partial S} \tilde{\mu}_{n-1}$$

the surface area of S with respect to μ and the quotient metric (see Section 3 for more details). By continuity in the radial topology, this extends to non smooth star bodies as well.

Let us now formulate our main theorem.

Theorem 1. *Let μ be a definition of volume, and let V be an n-dimensional normed space with unit ball B.*

i) *There exists a star body $\tilde{\mathbb{I}}_\mu B \subset V$ such that*

$$\tilde{A}_\mu(S) = n\tilde{V}(S[n-1], \tilde{\mathbb{I}}_\mu B), \quad S \in \mathcal{S}^\infty(V).$$

Here $\tilde{V}$ denotes the dual mixed volume [17]. We call $\tilde{\mathbb{I}}_\mu B$ the dual isoperimetrix.

ii) *Dual isoperimetric problem: Among all star bodies of a given volume, a dilate of the dual isoperimetrix has* maximal *surface area.*

iii) *Suppose that the dual definition of volume μ^* is convex. Then the usual isoperimetrix for the dual definition of volume μ^* and the dual isoperimetrix are related by*

$$\tilde{\mathbb{I}}_\mu B = \left(\mathbb{I}_{\mu^*} B^\circ\right)^\circ.$$

Corollary 1.1. *The dual isoperimetrix for Busemann's definition of volume μ^b is*

$$\tilde{\mathbb{I}}_{\mu^b}(B) = \omega_{n-1}\Pi^\circ(B),$$

while for the Holmes-Thompson volume μ^{ht}, we have

$$\tilde{\mathbb{I}}_{\mu^{ht}}(B) = \frac{1}{\omega_{n-1}}I(B^\circ).$$

Our second main theorem is an affinely invariant inequality. A survey over affinely invariant inequalities can be found in [19].

Theorem 2 (Surface area of the unit sphere). *Let (V, B) be an n-dimensional normed space. Then*

$$\tilde{A}_{\mu^b}(B) \leq n\omega_n, \tag{1}$$

Equality is attained precisely for centered ellipsoids.

In the two-dimensional case, this bound was conjectured by Faifman [9], who gave the non-optimal upper bound of 8. He also gave a lower bound of 4 and conjectured that $8 \log 2 \approx 5.5$ (which appears in the case of a square) is the optimal lower bound. Using John's ellipsoid, one can improve the lower bound in the two-dimensional case to $\sqrt{2}\pi \approx 4.4$, but we do not have a sharp lower bound.

As a corollary, we prove an upper bound for the quotient girth. Recall that the *girth* of a normed space is the length of the shortest symmetric curve on the unit sphere, measured with the Finsler metric induced by the norm. Analogously, the quotient girth is the length of the shortest symmetric curve on the unit sphere, measured with the quotient Finsler metric.

Corollary 1.2. *In any dimension, the quotient girth is bounded from above by 2π, with equality precisely for ellipsoids.*

2. The isoperimetrix in the Brunn-Minkowski theory

In this section, we recall some classical notions from convex geometry and Finsler geometry. References for this section are [3, 10, 28].

Definition 2.1 (Definition of volume). *A definition of volume assigns to each normed vector space $(V, \|\cdot\|)$ of dimension $n < \infty$ a norm μ_V on the one-dimensional space $\Lambda^n V$ such that the following three conditions are satisfied:*

i) *If V is Euclidean, then μ_V is the norm induced by usual Lebesgue measure.*

ii) *If $f : (V, \|\cdot\|) \to (W, \|\cdot\|)$ is a linear map that does not increase distances, then the induced map $\Lambda^n f : (\Lambda^n V, \mu_V) \to (\Lambda^n W, \mu_W)$ does not increase distances.*

iii) *The map $(V, \|\cdot\|) \mapsto (\Lambda^n V, \mu_V)$ is continuous with respect to Banach-Mazur distance.*

If B is the unit ball of V, we also write (V, B) instead of $(V, \|\cdot\|)$.

We will often need an alternative description of a definition of volume. Let μ be a definition of volume and (V, B) a normed space. Set $\mathcal{V}(B) := \int_B \mu$. Then $\mathcal{V}$ is a functional on the space of compact convex, centrally symmetric bodies with the origin in their interior. It is continuous, invariant under invertible linear maps and satisfies the following monotonicity condition: whenever $B_1 \subset B_2$ belong to the same n-dimensional vector space, then

$$\frac{\mathcal{V}(B_1)}{\mathcal{L}^n B_1} \geq \frac{\mathcal{V}(B_2)}{\mathcal{L}^n B_2}.$$

Here $\mathcal{L}^n$ is any choice of Lebesgue measure. Conversely, any continuous invariant functional with this monotonicity property comes from a definition of volume.

Before presenting some examples, we have to recall some notions from convex geometry. Given a compact convex body $K \subset V$, the support function is defined by

$$h_K(\xi) := \sup_{x \in K} \xi(x), \quad \xi \in V^*.$$

This function is positively 1-homogeneous and convex. Conversely, any positively 1-homogeneous and convex function is the support function of a unique compact convex body.

The polar body is defined by

$$K^\circ := \{\xi \in V^* : \langle \xi, x \rangle \leq 1, \quad \forall x \in K\}.$$

The radial function of a compact convex set K containing the origin in its interior is given by

$$\rho_K(x) := \max\{\lambda \geq 0 : \lambda x \in K\}, \quad x \in V \setminus \{0\}.$$

We will make frequent use of the well-known identity

$$\rho_K(x) = \frac{1}{h_{K^\circ}(x)}. \tag{2}$$

Examples

i) The *Busemann volume* is the definition of volume such that $\mathcal{V}(B) = \omega_n$ for any n-dimensional unit ball B. Here ω_n is the Euclidean volume of the Euclidean unit ball.

ii) The *Holmes-Thompson volume* is defined by

$$\mathcal{V}(B) = \frac{1}{\omega_n} \operatorname{vp}(B),$$

where $\operatorname{vp}(B)$ denotes the volume product of B, i.e. the symplectic volume of $B \times B^\circ \subset V \times V^*$.

iii) Gromov's mass is defined by

$$\mathcal{V}(B) := \frac{2^n}{n!} \frac{\mathcal{L}^n(B)}{\sup_{C \subset B} \mathcal{L}^n(C)}.$$

Here $\mathcal{L}^n$ is any Lebesgue measure and C ranges over all cross-polytopes inscribed in B.

iv) Benson definition (or Gromov's mass*).

$$\mathcal{V}(B) := 2^n \frac{\mathcal{L}^n(B)}{\inf_{P \supset B} \mathcal{L}^n(P)}.$$

Here P ranges over all parallelotopes circumscribed to B.

v) Ivanov's definition of volume [12]

$$\mathcal{V}(B) := \omega_n \frac{\mathcal{L}^n(B)}{\mathcal{L}^n(E)},$$

where E is the maximal volume ellipsoid inscribed in B (i.e. the John ellipsoid).

vi) The dual of Ivanov's definition of volume

$$\mathcal{V}(B) := \omega_n \frac{\mathcal{L}^n(B)}{\mathcal{L}^n(E)},$$

where E is the minimal volume ellipsoid circumscribed to B (i.e. the Löwner ellipsoid).

If μ is a definition of volume and V an n-dimensional vector space, μ_{V^*} is a norm on the one-dimensional vector space $\Lambda^n V^*$, whose dual $(\mu_{V^*})^*$ is a norm on $\Lambda^n V$. This motivates the following definition.

Definition 2.2 (Dual of a definition of volume, [3]). *Let μ be a definition of volume. Then the dual definition of volume μ^* is given by*

$$(\mu^*)_V := (\mu_{V^*})^*.$$

In terms of the associated functionals $\mathcal{V}$ and $\mathcal{V}^*$, we have

$$\mathcal{V}^*(B) := \frac{\mathrm{vp}(B)}{\mathcal{V}(B^\circ)}.$$

Examples

i) Busemann's definition of volume and Holmes-Thompson's definition of volume are dual to each other.

ii) Gromov's mass (3) and Gromov's mass* (4) are dual to each other.

iii) Ivanov's definition of volume (5) is dual to the definition of volume (6).

Definition 2.3.   i) *Let V be an n-dimensional vector space. A k-density on V, where $0 \le k \le n$, is a map $\phi : \Lambda_s^k V \to \mathbb{R}$ such that $\phi(\lambda a) = |\lambda| \phi(a)$ for all $\lambda \in \mathbb{R}, a \in \Lambda_s^k V$. Here $\Lambda_s^k V$ is the cone of simple k-vectors.*

ii) *Let M be an n-dimensional manifold. A k-density on M, where $0 \le k \le n$, is a continuous function $\Phi : \Lambda_s^k(TM) \to \mathbb{R}$ such that the restriction to each tangent space $T_p M, p \in M$ is a k-density in the sense of (1).*

Definition 2.4. *Let μ be a definition of volume. If V is a normed vector space, the induced $(n-1)$-volume density μ_{n-1} on V is defined as follows. If $v_1 \wedge \ldots \wedge v_{n-1} \ne 0$, then*

$$\mu_{n-1}(v_1 \wedge \ldots \wedge v_{n-1}) := \mu_W(v_1 \wedge \ldots \wedge v_{n-1}),$$

where W is the linear span of these vectors with the induced norm. Otherwise $\mu_{n-1}(v_1 \wedge \ldots \wedge v_{n-1}) = 0$.

Definition 2.5. *A definition of volume μ is called convex if $\mu_{n-1} : \Lambda^{n-1} V \to \mathbb{R}$ is a norm for each normed space V.*

Example: Busemann's definition of volume is convex. This is equivalent to the convexity of the intersection body (see below). Holmes-Thompson volume is also convex. This is equivalent to the convexity of the projection body. Gromov's mass* is convex, while Gromov's mass is not convex [3]. Ivanov's definition of volume is convex ([12], Thm. 6.2), while its dual is not convex.

Given a compact hypersurface $M \subset V$, the norm on V induces a Finsler metric on M and we may integrate μ_{n-1} over M. We will be mostly interested in the case where $M = \partial K$ is the boundary of a smooth compact convex body. If K is any compact convex body, then the integral still exists thanks to Alexandrov's theorem [1]. We will write

$$A_\mu(K) := \int_{\partial K} \mu_{n-1}$$

and call $A_\mu(K)$ the surface area of K with respect to μ.

Proposition 2.6 ([3]). *A definition of volume μ is convex if and only if for each pair $K \subset L$, we have*

$$A_\mu(K) \leq A_\mu(L).$$

Definition 2.7 (Isoperimetrix). *Let μ be a convex definition of volume. Then, for each normed space (V, B) of dimension n there exists a unique centrally symmetric compact convex body $\mathbb{I}_\mu(B) \subset V$ such that for all compact convex bodies $K \subset V$*

$$A_\mu(K) = nV(K[n-1], \mathbb{I}_\mu(B)).$$

Here V denotes the mixed volume [23]. The body $\mathbb{I}_\mu(B)$ is called isoperimetrix.

Strictly speaking, the isoperimetrix depends on the choice of a volume form. However, the definition of volume μ gives us a canonical choice of Lebesgue measure on V which we will use in the following.

Let us recall the construction of the isoperimetrix. The function $\mu_{n-1} : \Lambda^{n-1}V \to \mathbb{R}$ is convex and 1-homogeneous by assumption. The volume form on V induces an isomorphism $\Lambda^{n-1}V \cong V^*$. We thus get a convex and 1-homogeneous function on V^*, which is the support function of the isoperimetrix.

Proposition 2.8 (Isoperimetric inequality). *Let B be the unit ball of a normed space and μ a convex definition of volume. Among all compact convex bodies K with given volume, the surface area with respect to μ is minimal precisely for a homothet of the isoperimetrix $\mathbb{I}_\mu(B)$. Equivalently, for each choice of Lebesgue measure, we have*

$$\frac{A_\mu(K)^n}{\mathcal{L}^n(K)^{n-1}} \geq \frac{A_\mu(\mathbb{I}_\mu(B))^n}{\mathcal{L}^n(\mathbb{I}_\mu(B))^{n-1}}.$$

Examples:

i) The isoperimetrix for Busemann's volume is (up to a constant) the polar of the intersection body:

$$\mathbb{I}_{\mu^b}(B) = \omega_{n-1} I^\circ B.$$

Let us briefly recall the definition of the intersection body, referring to [11, 13, 15, 18, 24, 28] for details, more information and generalizations.

Given a non-zero volume form $\Omega \in \Lambda^n V^*$ and $\xi \in V^*, \xi \neq 0$, we may write (in a non-unique way) $\Omega = \xi \wedge \Omega_\xi$ with $\Omega_\xi \in \Lambda^{n-1}V^*$. Then the restriction of Ω_ξ to $\ker \xi$ is a volume form, which does not depend on the

choice of Ω_ξ. The intersection body of a star body $S \subset V$ is the star body IS in V^* whose radial function is given by

$$\rho(IS, \xi) = \mathrm{vol}(S \cap \ker \xi, \Omega_\xi).$$

By a non-trivial result due to Busemann [8], the intersection body of a centrally symmetric convex body is convex.

Busemann and Petty [6] have shown that the Busemann surface area $A_{\mu^b}(B)$ of the unit sphere is maximal precisely if B is a parallelotope. No lower bound seems to be known, except in dimensions 2 and 3. We refer to [3] and [4] for more results in this direction.

ii) For the Holmes-Thompson definition of volume, the isoperimetrix is (up to a constant) the projection body of the polar:

$$\mathbb{I}_{\mu^{ht}}(B) = \frac{1}{\omega_{n-1}} \Pi(B^\circ).$$

Let us also recall briefly the definition of the projection body. Again, we have to refer to the literature for a deeper study of projection bodies [13, 14, 24, 25, 28].

If $v \in V, v \neq 0$, then $i_v\Omega := \Omega(v, \cdot)$ is a volume form on $V/\langle v \rangle$. Let $\pi_v : V \to V/\langle v \rangle$ be the projection map. The projection body ΠK of a compact convex body K is the compact convex body in V^* whose support function is given by

$$h(\Pi K, v) = \mathrm{vol}(\pi_v K, i_v\Omega), \quad v \in V \setminus \{0\}.$$

The projection body of a compact convex body is convex.

Let us recall a famous geometric inequality related to the projection body.

Theorem 2.9 (Petty's projection inequality, [21])**.** *Let $K \subset V$ be a compact convex body and $E \subset V$ an ellipsoid. Then*

$$\mathrm{vol}(K)^{n-1} \mathrm{vol}\, \Pi^\circ K \leq \mathrm{vol}(E)^{n-1} \mathrm{vol}\, \Pi^\circ E$$

with equality precisely for ellipsoids.

Holmes and Thompson have shown that the Holmes-Thompson surface area $A_{\mu^{ht}}(B)$ is the same as the Holmes-Thompson surface area $A_{\mu^{ht}}(B^\circ)$ (computed in the dual normed space (V^*, B°)). This result was later put in a symplectic geometry framework by Alvarez [2], who reproved their result and showed that dual spheres have the same girth (the girth is the length of the shortest symmetric geodesic). This confirmed a conjecture by Schäffer [22].

3. THE ISOPERIMETRIX IN THE DUAL BRUNN-MINKOWSKI THEORY

The natural setting for the dual Brunn-Minkowski theory is that of star bodies (instead of convex bodies) and radial addition (instead of Minkowski addition). We let $S_1 \tilde{+} S_2$ denote the radial addition of S_1 and S_2, i.e.

$$\rho_{S_1 \tilde{+} S_2} = \rho_{S_1} + \rho_{S_2}, \quad S_1, S_2 \in \mathcal{S}(V),$$

where ρ denotes the radial function and $\mathcal{S}(V)$ is the space of star bodies. The natural topology on $\mathcal{S}(V)$ is the radial topology. The dense subspace of star bodied with smooth radial function is denoted by $\mathcal{S}^\infty(V)$.

Given an n-dimensional vector space V with a volume form $\Omega \in \Lambda^n V^*$ and corresponding Lebesgue measure $\mathcal{L}^n$, there is a unique functional

$$\tilde{V} : \mathcal{S}(V)^n \to \mathbb{R}$$

called *dual mixed volume*, which is symmetric, multi-linear (with respect to radial addition), continuous with respect to the radial topology and which satisfies $\tilde{V}(S, \ldots, S) = \mathrm{vol}(S)$. It was introduced by Lutwak [17].

Explicitly, we have for $S_1, \ldots, S_n \in \mathcal{S}(V)$

$$\tilde{V}(S_1, \ldots, S_n) = \frac{1}{n} \int_\Sigma \rho_{S_1} \cdots \rho_{S_n} i_v \Omega,$$

where Σ is any $n-1$-submanifold in the same homology class as the sphere and $(i_v \Omega)|_v := \Omega(v, \cdot)$ is an $(n-1)$-form on V. We refer to [30] for more information on such *contour integrals*. In particular, taking $\Sigma := \partial S$,

$$\tilde{V}(S[n-1], T) = \frac{1}{n} \int_{\partial S} \rho_T i_v \Omega.$$

Theorem 3.1 (Dual Minkowski inequalities). *For $K, L \in \mathcal{S}(V)$ we have*

$$\tilde{V}(K, \ldots, K, L)^n \leq \mathrm{vol}(K)^{n-1} \mathrm{vol}(L).$$

Let $S \in \mathcal{S}^\infty(V)$. Each tangent space $T_p \partial S$ may be considered as a subspace of V. But it can also be considered as the quotient

$$T_p \partial S = V/\langle p \rangle.$$

with the quotient norm. Explicitly,

$$\|v\| = \inf_{t \in \mathbb{R}} \|v + tp\|, \quad v \in T_p \partial S,$$

which is not larger than the induced norm.

Definition 3.2. *Let μ be a definition of volume. If (V, B) is a normed vector space and $M \subset V$ an $(n-1)$-dimensional submanifold such that $p \notin T_p M$ for all $p \in M$, there is an induced $(n-1)$-volume density $\tilde{\mu}_{n-1}$ on M defined as follows. Let $p \in M$ and $v_1, \ldots, v_{n-1} \in T_p M$. If $v_1 \wedge \ldots \wedge v_{n-1} \neq 0$, then*

$$\tilde{\mu}_{n-1}(v_1 \wedge \ldots \wedge v_{n-1}) := \mu_{T_p M}(v_1 \wedge \ldots \wedge v_{n-1}),$$

where $T_p M = V/\langle p \rangle$ is endowed with the quotient norm. Otherwise $\mu_{n-1}(v_1 \wedge \ldots \wedge v_{n-1}) = 0$.

We may integrate $\tilde{\mu}_{n-1}$ over M. In the particular case where M is the boundary of a star-shaped set, this yields the following definition.

Definition 3.3. *Let μ be a definition of volume and $S \in \mathcal{S}^\infty(V)$. We call*

$$\tilde{A}_\mu(S) := \int_{\partial S} \tilde{\mu}_{n-1}$$

the dual surface area of S with respect to the definition of volume μ.

It will follow from Equation (4) below that this definition extends by continuity to all star bodies.

The quotient metric on the boundary of a unit ball was first studied by Faifman [9], who showed an analogue of Alvarez' result mentioned in the last section. More precisely, he proved that the girth, the length spectrum and Holmes-Thompson volume of the unit sphere and of the polar unit sphere are the same.

Proof of Theorem 1. In the following, the pairing between an element of a vector space and an element of the dual vector space will be denoted by $\langle \cdot, \cdot \rangle$.

Let V be an n-dimensional normed space and $\Omega \in \Lambda^n V^*, \Omega \neq 0$ a volume form. There is an isomorphism

$$\tau : V \to \Lambda^{n-1} V^* = (\Lambda^{n-1} V)^*$$
$$w \mapsto [a \mapsto \Omega(a \wedge w)].$$

Let $v \in V, v \neq 0$. Clearly $\tau(v)$ vanishes on $(n-1)$-vectors of the form $a = v \wedge u, u \in \Lambda^{n-2} V$, hence it belongs to $\Lambda^{n-1} \ker v$, where $\ker v := \{\eta \in V^* : \langle \eta, v \rangle = 0\}$. Therefore $\langle v \rangle$ is mapped isomorphically to $\Lambda^{n-1} \ker v$.

Let $W := (V/\langle v \rangle, \pi_v B)$. We claim that there is a duality of normed spaces

$$W^* = (V/\langle v \rangle, \pi_v B)^* = (\ker v, B^\circ \cap \ker v).$$

Indeed, the dual of the projection $\pi_v : V \twoheadrightarrow V/\langle v \rangle$ is the inclusion $\pi_v^* : \ker v \hookrightarrow V^*$. Hence

$$(\pi_v B)^\circ = \{\xi \in \ker v | \langle \xi, \pi_v p \rangle \leq 1 \forall p \in B\}$$
$$= \{\xi \in \ker v | \langle \pi_v^* \xi, p \rangle \leq 1 \forall p \in B\}$$
$$= \{\xi \in \ker v | \pi_v^* \xi \in B^\circ\}$$
$$= B^\circ \cap \ker v.$$

We define $h : V \to \mathbb{R}$ by $h(v) := \mu_{n-1}^*(\tau(v))$. Clearly h is 1-homogeneous and positive. Since μ is the dual definition of volume of μ^*, we have for $a \in \Lambda^{n-1} W$

$$\tilde{\mu}_{n-1}(a) = \mu_W(a)$$
$$= \sup\{\langle a, b \rangle | b \in \Lambda^{n-1} W^*, \mu_{W^*}^*(b) \leq 1\}$$
$$= \sup\{\langle a, \tau(w) \rangle | w \in \langle v \rangle, \mu_{n-1}^*(\tau(w)) \leq 1\}$$
$$= \sup\{\Omega(a \wedge w) | w \in \langle v \rangle, h(w) \leq 1\}$$
$$= \frac{1}{h(v)} |\Omega(a \wedge v)|$$
$$= \left| i_{\frac{v}{h(v)}} \Omega(a) \right|.$$

Define the star body T by

$$\rho_T(v) = \frac{1}{h(v)}. \tag{3}$$

It follows that

$$\tilde{A}_\mu(S) = \int_{\partial S} \tilde{\mu}_{n-1} = \int_{\partial S} \rho_T(v) i_v \Omega = n \tilde{V}(S[n-1], T). \tag{4}$$

With $\tilde{\mathbb{I}}_\mu B := T$, the first statement of the theorem follows.

By the dual Minkowski inequality (Theorem 3.1), we find

$$\tilde{A}_\mu(S) \leq n \operatorname{vol}(S)^{\frac{n-1}{n}} \operatorname{vol}(\tilde{\mathbb{I}}_\mu B)^{\frac{1}{n}}$$

with equality if and only if S and $\tilde{\mathbb{I}}_\mu B$ are dilates of each other. This shows the second statement of the theorem.

Finally, suppose that μ^* is convex in the sense of Definition 2.5. By the construction of the isoperimetrix, the function h defined above is the support function

of $\mathbb{I}_{\mu^*}B^\circ$. From (2) and (3) it follows that $\tilde{\mathbb{I}}_\mu B$ is the polar of $\mathbb{I}_{\mu^*}B^\circ$, which finishes the proof. $\qquad\square$

Proof of Theorem 2. Let E be an ellipsoid. By the dual Minkowski inequality (Theorem 3.1) and Petty's projection inequality (Theorem 2.9) we have

$$
\begin{aligned}
\tilde{A}_{\mu^b}(B) &= n\omega_{n-1}\tilde{V}(B[n-1], \Pi^\circ B)\\
&\leq n\omega_{n-1}(\mathrm{vol}\,B)^{\frac{n-1}{n}}(\mathrm{vol}\,\Pi^\circ B)^{\frac{1}{n}}\\
&\leq n\omega_{n-1}(\mathrm{vol}\,E)^{\frac{n-1}{n}}(\mathrm{vol}\,\Pi^\circ E)^{\frac{1}{n}}\\
&= n\omega_{n-1}\tilde{V}(E[n-1], \Pi^\circ E)\\
&= \tilde{A}_\mu(E)\\
&= n\omega_n.
\end{aligned}
$$

The equality case follows from the fact that equality in Petty's projection inequality is attained for (not necessarily centered) ellipsoids and that $B = -B$. $\quad\square$

Proof of Corollary 1.2. Clearly the quotient girth of B is not larger than the quotient girth of any central two-dimensional section, which is bounded by 2π by the above theorem. If B is not an ellipsoid, then there exists a two-dimensional central section which is not an ellipsoid (see [10], Thm. 7.1.5) and whose quotient girth is strictly smaller than 2π. $\qquad\square$

References

[1] A. D. Alexandroff. Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. *Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser.*, 6:3–35, 1939.

[2] J. C. Álvarez Paiva. Dual spheres have the same girth. *Amer. J. Math.*, 128(2):361–371, 2006.

[3] J. C. Álvarez Paiva and A. C. Thompson. Volumes on normed and Finsler spaces. In *A sampler of Riemann-Finsler geometry*, volume 50 of *Math. Sci. Res. Inst. Publ.*, pages 1–48. Cambridge Univ. Press, Cambridge, 2004.

[4] Juan Carlos Álvarez Paiva and Anthony Thompson. On the perimeter and area of the unit disc. *Amer. Math. Monthly*, 112(2):141–154, 2005.

[5] Luigi Ambrosio and Bernd Kirchheim. Currents in metric spaces. *Acta Math.*, 185(1):1–80, 2000.

[6] H. Busemann and C. M. Petty. Problems on convex bodies. *Math. Scand.*, 4:88–94, 1956.

[7] Herbert Busemann. The isoperimetric problem for Minkowski area. *Amer. J. Math.*, 71:743–762, 1949.

[8] Herbert Busemann. A theorem on convex bodies of the Brunn-Minkowski type. *Proc. Nat. Acad. Sci. U. S. A.*, 35:27–31, 1949.

[9] Dmitry Faifman. An extension of Schäffer's dual girth conjecture to Grassmannians. *J. Differential Geom.*, 92(1):201–220, 2012.

[10] Richard J. Gardner. *Geometric tomography*, volume 58 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, second edition, 2006.

[11] Christoph Haberl. L_p intersection bodies. *Adv. Math.*, 217(6):2599–2624, 2008.

[12] S. V. Ivanov. Volumes and areas of Lipschitz metrics. *Algebra i Analiz*, 20(3):74–111, 2008.

[13] Alexander Koldobsky. *Fourier analysis in convex geometry*, volume 116 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005.

[14] Monika Ludwig. Projection bodies and valuations. *Adv. Math.*, 172(2):158–168, 2002.

[15] Monika Ludwig. Intersection bodies and valuations. *Amer. J. Math.*, 128(6):1409–1428, 2006.

[16] Monika Ludwig. Minkowski areas and valuations. *J. Differential Geom.*, 86(1):133–161, 2010.

[17] Erwin Lutwak. Dual mixed volumes. *Pacific J. Math.*, 58(2):531–538, 1975.

[18] Erwin Lutwak. Intersection bodies and dual mixed volumes. *Adv. in Math.*, 71(2):232–261, 1988.

[19] Erwin Lutwak. Selected affine isoperimetric inequalities. In *Handbook of convex geometry, Vol. A, B*, pages 151–176. North-Holland, Amsterdam, 1993.

[20] Frank Morgan. Minimal surfaces, crystals, shortest networks, and undergraduate research. *Math. Intelligencer*, 14(3):37–44, 1992.

[21] Clinton M. Petty. Isoperimetric problems. In *Proceedings of the Conference on Convexity and Combinatorial Geometry (Univ. Oklahoma, Norman, Okla., 1971)*, pages 26–41. Dept. Math., Univ. Oklahoma, Norman, Okla., 1971.

[22] Juan Jorge Schäffer. *Geometry of spheres in normed spaces*. Marcel Dekker Inc., New York, 1976. Lecture Notes in Pure and Applied Mathematics, No. 20.

[23] Rolf Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993.

[24] Franz E. Schuster. Valuations and Busemann-Petty type problems. *Adv. Math.*, 219(1):344–368, 2008.

[25] Franz E. Schuster and Thomas Wannerer. $GL(n)$ contravariant Minkowski valuations. *Trans. Amer. Math. Soc.*, 364(2):815–826, 2012.

[26] Jean E. Taylor. Unique structure of solutions to a class of nonelliptic variational problems. In *Differential geometry (Proc. Sympos. Pure. Math., Vol. XXVII, Stanford Univ., Stanford, Calif., 1973), Part 1*, pages 419–427. Amer. Math. Soc., Providence, R.I., 1975.

[27] Jean E. Taylor. Crystalline variational problems. *Bull. Amer. Math. Soc.*, 84(4):568–588, 1978.

[28] Anthony C. Thompson. *Minkowski geometry*, volume 63 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1996.

[29] G. Wulff. Zur Frage der Geschwindigkeit des Wachstums und der Auflösung der Krystallflächen. *Z. Krystallogr. Mineral.*, 34:449–530, 1901.

[30] Deane Yang. Affine geometry using the homogeneous contour integral. In *Proceedings of the 14th International Workshop on Differential Geometry and the 3rd KNUGRG-OCAMI Differential Geometry Workshop [Volume 14]*, pages 1–12. Natl. Inst. Math. Sci. (NIMS), Taejŏn, 2010.

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